

A note on nowhere-zero 3-flow and Z_3 -connectivity

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ABSTRACT. There are many major open problems in integer flow theory, such as Tutte's 3-flow conjecture that every 4-edge-connected graph admits a nowhere-zero 3-flow, Jaeger et al.'s conjecture that every 5-edge-connected graph is Z_3 -connected and Kochol's conjecture that every bridgeless graph with at most three 3-edge-cuts admits a nowhere-zero 3-flow (an equivalent version of 3-flow conjecture). Thomassen proved that every 8-edge-connected graph is Z_3 -connected and therefore admits a nowhere-zero 3-flow. Furthermore, Lovász, Thomassen, Wu and Zhang improved Thomassen's result to 6-edge-connected graphs. In this paper, we prove that: (1) Every 4-edge-connected graph with at most seven 5-edge-cuts admits a nowhere-zero 3-flow. (2) Every bridgeless graph containing no 5-edge-cuts but at most three 3-edge-cuts admits a nowhere-zero 3-flow. (3) Every 5-edge-connected graph with at most five 5-edge-cuts is Z_3 -connected. Our main theorems are partial results to Tutte's 3-flow conjecture, Kochol's conjecture and Jaeger et al.'s conjecture, respectively.

Keywords: Integer flow, Nowhere-zero 3-flow, Z_3 -Connected, Modulo 3-orientation, Edge-cuts.

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1 Introduction

All graphs considered in this paper are loopless, but allowed to have multiple edges. A graph G is called k -edge-connected, if $G - S$ is connected for each edge set S with $|S| < k$. Let X, Y be two disjoint subsets of $V(G)$. Let $\partial_G(X, Y)$ be the set of edges of G with one end in X and the other in Y . In particular, if $Y = \overline{X}$, we simply write $\partial_G(X)$ for $\partial_G(X, Y)$, which is the *edge-cut* of G associated with X . The edge set $C = \partial_G(X)$ is called a k -edge-cut if $|\partial_G(X)| = k$. If X is nontrivial, we use G/X to denote the graph obtained from G by replacing X by a single vertex x that is incident with all the edges in $\partial_G(X)$.

Let D be an orientation of $E(G)$. The *out-cut* of D associated with X , denoted by $\partial_D^+(X)$, is the set of arcs of D whose tails lie in X . Analogously, the *in-cut* of D associated with X , denoted by $\partial_D^-(X)$, is the set of arcs of D whose heads lie in X . We refer to $|\partial_D^+(X)|$ and $|\partial_D^-(X)|$ as the out-degree and in-degree of X , and denote these quantities by $d_D^+(X)$ and $d_D^-(X)$, respectively.

Definition 1.1. (1) An orientation D of $E(G)$ is called a modulo 3-orientation if

$$d_D^+(v) - d_D^-(v) \equiv 0 \pmod{3}$$

for every vertex $v \in V(G)$.

(2) A pair (D, f) is called a nowhere-zero 3-flow of G if D is an orientation of $E(G)$ and f is a function from $E(G)$ to $\{\pm 1, \pm 2\}$, such that

$$\sum_{e \in \partial_D^+(v)} f(e) = \sum_{e \in \partial_D^-(v)} f(e)$$

for every vertex $v \in V(G)$.

The 3-flow conjecture, proposed by Tutte as a dual version of Grötzsch's 3-color theorem for planar graphs, may be one of the most major open problems in integer flow theory.

Conjecture 1.2. (3-Flow conjecture, Tutte [9]) Every 4-edge-connected graph admits a nowhere-zero 3-flow.

Kochol proved that Tutte's 3-flow conjecture is equivalent to the following two conjectures.

Conjecture 1.3. (Kochol [4]) Every 5-edge-connected graph admits a nowhere-zero 3-flow.

Conjecture 1.4. (Kochol [5]) Every bridgeless graph with at most three 3-edge-cuts

admits a nowhere-zero 3-flow.

A weakened version of Conjecture 1.2, the so-called weak 3-flow conjecture, was proposed by Jaeger.

Conjecture 1.5. (Weak 3-flow conjecture, Jaeger [2]) *There is a natural number h such that every h -edge-connected graph admits a nowhere-zero 3-flow.*

Lai and Zhang [6] and Alon et al. [1] gave partial results on Conjectures 1.2 and 1.5.

Theorem 1.6. (Lai and Zhang [6]) *Every $4\lceil\log_2 n_0\rceil$ -edge-connected graph with at most n_0 odd-degree vertices admits a nowhere-zero 3-flow.*

Theorem 1.7. (Alon, Linial and Meshulam [1]) *Every $2\lceil\log_2 n\rceil$ -edge-connected graph with n vertices admits a nowhere-zero 3-flow.*

Recently, Thomassen [8] confirmed weak 3-flow conjecture. He proved

Theorem 1.8. (Thomassen [8]) *Every 8-edge-connected graph is Z_3 -connected and therefore admits a nowhere-zero 3-flow.*

Thomassen's method was further refined by Lovász, Thomassen, Wu and Zhang [7] to obtain the following theorem.

Theorem 1.9. (Lovász, Thomassen, Wu and Zhang [7]) *Every 6-edge-connected graph is Z_3 -connected and therefore admits a nowhere-zero 3-flow.*

For more results on Tutte's 3-flow conjecture, we refer the reader to the introduction part of [7] and the book written by Zhang [11].

In this paper, we will give the following conjecture which is equivalent to Tutte's 3-flow conjecture.

Conjecture 1.10. *Every 5-edge-connected graph with minimum degree at least 6 has a nowhere-zero 3-flow.*

To prove the equivalence of Conjectures 1.2 and 1.10, the following lemma is needed.

Lemma 1.11. (Tutte [10]) *Let $F(G, k)$ be the number of nowhere-zero k -flows of G . Then $F(G, k) = F(G/e, k) - F(G \setminus e, k)$ if e is not a loop of G .*

Proposition. *Conjectures 1.2 and 1.10 are equivalent.*

Proof. It is obvious that Conjecture 1.2 implies Conjecture 1.3, and Conjecture 1.3 implies

Conjecture 1.10. Now we prove that Conjecture 1.10 can imply Conjecture 1.3. Let G be a 5-edge-connected graph. Let G' be the graph obtained from G by gluing $|V(G)|$ disjoint copies of K_7 , such that for each such copy H_i , $|V(H_i) \cap V(G)| = 1$ ($i = 1, 2, \dots, |V(G)|$). Then G' is 5-edge-connected and its minimum degree is at least 6, and thus has a nowhere-zero 3-flow. By Lemma 1.11, G has a nowhere-zero 3-flow. Therefore Conjecture 1.10 implies Conjecture 1.3. Note that Conjecture 1.2 is equivalent to Conjecture 1.3. This completes the proof. ■

Our first main result is the following theorem.

Theorem 1.12. *Let G be a bridgeless graph and let $P = \{C = \partial_G(X) : |C| = 3, X \subset V(G)\}$ and $Q = \{C = \partial_G(X) : |C| = 5, X \subset V(G)\}$. If $2|P| + |Q| \leq 7$, then G has a modulo 3-orientation (and therefore has a nowhere-zero 3-flow).*

As corollaries of Theorem 1.12, we obtain Theorems 1.13 and 1.14.

Theorem 1.13. *Every 4-edge-connected graph with at most seven 5-edge-cuts admits a nowhere-zero 3-flow.*

Theorem 1.14. *Every bridgeless graph containing no 5-edge-cuts but at most three 3-edge-cuts admits a nowhere-zero 3-flow.*

Remark. *The number of 3-edge-cuts in Theorem 1.14 can not be improved from three to four since K_4 or any graph contractible to K_4 has no nowhere-zero 3-flow.*

Theorems 1.13 and 1.14 partially confirm Conjectures 1.2 and 1.4, respectively.

Definition 1.15. (1) A mapping $\beta_G : V(G) \mapsto Z_k$ is called a Z_k -boundary of G if

$$\sum_{v \in V(G)} \beta_G(v) \equiv 0 \pmod{k}$$

(2) A graph G is called Z_k -connected, if for every Z_k -boundary β_G , there is an orientation D_{β_G} and a function $f_{\beta_G} : E(G) \mapsto Z_k - \{0\}$, such that

$$\sum_{e \in \partial_{D_{\beta_G}}^+(v)} f_{\beta_G}(e) - \sum_{e \in \partial_{D_{\beta_G}}^-(v)} f_{\beta_G}(e) \equiv \beta_G(v) \pmod{k}$$

for every vertex $v \in V(G)$.

Jaeger, Linial, Payan and Tarsi [3] conjectured that

Conjecture 1.16. (Jaeger, Linial, Payan and Tarsi [3]) *Every 5-edge-connected graph is Z_3 -connected.*

By applying a similar argument in the proof of Theorem 1.12, we could obtain the second main result which is a partial result to Conjecture 1.16.

Theorem 1.17. *Every 5-edge-connected graph with at most five 5-edge-cuts is Z_3 -connected.*

In the next section, some necessary preliminaries will be given. In Sections 3 and 4, proofs of Theorems 1.12 and 1.17 will be given, respectively.

2 Preliminaries

In this section, we will give additional but necessary notations and definitions, and then give some useful lemmas.

Definition 2.1 *Let β_G be a Z_3 -boundary of G . An orientation D of G is called a β_G -orientation if*

$$d_D^+(v) - d_D^-(v) \equiv \beta_G(v) \pmod{3}$$

for every vertex $v \in V(G)$.

Let G be a graph and A be a vertex subset of G . The *degree* of A , denoted by $d_G(A)$, is the number of edges with precisely one end in A . Moreover if $A = \{x\}$, we simply write $d_G(x)$.

Let G be a graph and β_G be a Z_3 -boundary of G . Define a mapping $\tau_G : V(G) \mapsto \{0, \pm 1, \pm 2, \pm 3\}$ such that, for each vertex $x \in V(G)$,

$$\tau_G(x) \equiv \begin{cases} \beta_G(x) & (\text{mod } 3) \\ d_G(x) & (\text{mod } 2). \end{cases}$$

Now, the mapping τ_G can be further extended to any nonempty vertex subset A as follows:

$$\tau_G(A) \equiv \begin{cases} \beta_G(A) & (\text{mod } 3) \\ d_G(A) & (\text{mod } 2). \end{cases}$$

where $\beta_G(A) \equiv \sum_{x \in A} \beta_G(x) \in \{0, 1, 2\} \pmod{3}$.

Proposition 2.2 *Let G be a graph and A be a vertex subset of G .*

(1) *If $d_G(A) \leq 5$, then $d_G(A) \leq 4 + |\tau_G(A)|$.*

(2) *If $d_G(A) \geq 6$, then $d_G(A) \geq 4 + |\tau_G(A)|$.*

Proposition 2.2 follows from the fact that $|\tau_G(A)| \leq 3$ and $d_G(A) - |\tau_G(A)|$ is even.

Lemma 2.3 (Tutte [9]) *Let G be a graph.*

(1) *G has a nowhere-zero 3-flow if and only if G has a modulo 3-orientation.*

(2) *G has a nowhere-zero 3-flow if and only if G has a β_G -orientation with $\beta_G = 0$.*

The following lemma is Theorem 3.1 in [7] by Lovász et al. This lemma will play the main role in our proofs.

Lemma 2.4 (Lovász, Thomassen, Wu and Zhang [7]) Let G be a graph, β_G be a \mathbb{Z}_3 -boundary of G , and let $z_0 \in V(G)$ and D_{z_0} be a pre-orientation of $E(z_0)$ of all edges incident with z_0 . Assume that

- (i) $|V(G)| \geq 3$.
- (ii) $d_G(z_0) \leq 4 + |\tau_G(z_0)|$ and $d_{D_{z_0}}^+(z_0) - d_{D_{z_0}}^-(z_0) \equiv \beta_G(z_0) \pmod{3}$, and
- (iii) $d_G(A) \geq 4 + |\tau_G(A)|$ for each nonempty vertex subset A not containing z_0 with $|V(G) \setminus A| > 1$.

Then the pre-orientation D_{z_0} of $E(z_0)$ can be extended to an orientation D of the entire graph G , that is, for every vertex x of G ,

$$d_D^+(x) - d_D^-(x) \equiv \beta_G(x) \pmod{3}.$$

3 Proof of Theorem 1.12

If not, suppose that G is a counterexample, such that $|V(G)| + |E(G)|$ is as small as possible. Let $P' = \{x \in V(G) : d_G(x) = 3\}$ and $Q' = \{x \in V(G) : d_G(x) = 5\}$.

Claim 1. $|V(G)| \geq 3$.

Proof. If $|V(G)| = 1$, then G has a nowhere-zero 3-flow, a contradiction. If $|V(G)| = 2$, let $V(G) = \{x, y\}$, then all the edges of G are all between x and y . Since G is bridgeless, $|E(G)| \geq 2$. Let a be the integer in $\{0, 1, 2\}$ such that $a \equiv |E(G)| \pmod{3}$. Orient a edges from x to y and the remaining $|E(G)| - a$ edges from y to x . Clearly, the resulting orientation is a modulo 3-orientation of G , a contradiction. Therefore $|V(G)| \geq 3$. ■

Claim 2. G is 3-edge-connected, and G has no nontrivial 3-edge-cuts.

Proof. If G has a vertex x of degree 2, then suppose that $xx_1, xx_2 \in E(G)$. By the minimality of G , $(G - \{xx_1, xx_2\}) \cup \{x_1x_2\}$ has a nowhere-zero 3-flow f' . However, f' can be extended to a nowhere-zero 3-flow f of G , a contradiction. If G has a nontrivial k -edge-cut ($k = 2, 3$), then contract one side and find a mod 3-orientation by the minimality of G . Merge such two mod 3-orientations and we will get one for G , a contradiction. ■

Claim 3. For any $U \subset V(G)$, if $d_G(U) \leq 5$ and $|U| \geq 2$, then $U \cap (P' \cup Q') \neq \emptyset$.

Proof. If not, choose U to be a minimal one such that: for any $A \subset U$ with $2 \leq |A| < |U|$, we have $d_G(A) \geq 6$.

By the minimality of G , G/U has a modulo 3-orientation D' which is a partial modulo 3-orientation of G , such that $d_{D'}^+(x) \equiv d_{D'}^-(x) \pmod{3}$ for each $x \in V(G) \setminus U$.

Let G' be a graph obtained from G by contracting $V(G) \setminus U$ as z_0 and let $\beta_{G'} = 0$.

- (i) Since $V(G') = U + z_0$, $|V(G')| = |U| + 1 \geq 3$.
- (ii) Since $d_{G'}(z_0) = d_G(U) \leq 5$, by Proposition 2.2 (1), $d_{G'}(z_0) \leq 4 + |\tau_{G'}(z_0)|$.

(iii) By the assumption and minimality of U , we have that $\forall A \subset U$, $d_G(A) \neq 5$ and $d_G(A) \neq 3$. If $d_G(A) = 4$, then $d_{G'}(A) = d_G(A) = 4$ and $\tau_{G'}(A) = \beta_{G'}(A) = \beta_G(A) = 0$. Thus $d_{G'}(A) = 4 = 4 + |\tau_{G'}(A)|$. If $d_G(A) \geq 6$, then by Proposition 2.2 (2), $d_{G'}(A) = d_G(A) \geq 4 + |\tau_{G'}(A)|$.

By Lemma 2.4, we could see that the pre-orientation of $E'(z_0)$ of all edges incident with z_0 can be extended to a $\beta_{G'}$ -orientation of G' . Then G has a modulo 3-orientation, which is a contradiction. ■

Let G'_1 be a graph obtained from G by adding a new vertex z_0 and $2|P'| + |Q'|$ edges between z_0 and $P' \cup Q'$, such that:

- (i) For each vertex $v \in P'$, we add two arcs with the same direction between it and z_0 ; and
- (ii) For each vertex $v \in Q'$, we add one arc between it and z_0 .

If $2|P'| + |Q'| \leq 5$, then all added arcs could be from z_0 to $P' \cup Q'$. Define $\beta_{G'_1}$ as follows:

- (1) $\beta_{G'_1}(x) = 0$ if $x \notin (P' \cup Q') + z_0$;
- (2) $\beta_{G'_1}(x) = 1$ if $x \in P'$;
- (3) $\beta_{G'_1}(x) = 2$ if $x \in Q'$;
- (4) $\beta_{G'_1}(z_0) \equiv 2|P'| + |Q'| \pmod{3}$ and $\beta_{G'_1}(z_0) \in \{0, 1, 2\}$.

If $2|P'| + |Q'| = 6$ or 7 , in this case, if $|P'| \neq 0$, choose one vertex $v \in P'$, such that the two arcs with ends z_0 and v are from v to z_0 , the other arcs incident with z_0 are all directed from z_0 . If $|P'| = 0$, then two arcs are from Q' to z_0 , the others verse. Define $\beta_{G'_1}$ as follows:

- (1) $\beta_{G'_1}(x) = 0$ if $x \notin (P' \cup Q') + z_0$;
- (2) $\beta_{G'_1}(x) = 2$ if $x \in Q'$ and the arc (z_0, x) exists or $x \in P'$ and the two arcs with ends z_0 and x are from x to z_0 ;
- (3) $\beta_{G'_1}(x) = 1$ if $x \in Q'$ and the arc (x, z_0) exists or $x \in P'$ and the two arcs with ends z_0 and x are from z_0 to x ;
- (4) $\beta_{G'_1}(z_0) \equiv (2|P'| + |Q'| - 2) \pmod{3}$.

Now $d_{G'_1}(z_0) \leq 4 + |\tau_{G'_1}(z_0)|$ and $|V(G'_1)| = |V(G)| + 1 \geq 4$. We claim that: $d_{G'_1}(A) \geq 4 + |\tau_{G'_1}(A)|$, for each nonempty vertex subset A not containing z_0 with $|V(G'_1) \setminus A| > 1$.

If $A \cap (P' \cup Q') = \emptyset$, then by Claim 3, $d_G(A) = 4$ or $d_G(A) \geq 6$. In each case we could get that $d_{G'_1}(A) = d_G(A) \geq 4 + |\tau_{G'_1}(A)|$.

If $A \cap (P' \cup Q') \neq \emptyset$, then by Claim 2, $d_{G'_1}(A) \geq 5$. If $d_{G'_1}(A) = 5$, then $d_G(A) = 3$ or 4 and $|A \cap (P' \cup Q')| = 1$, and it follows that $\beta_{G'_1}(A) = 1$ or 2 , and $|\tau_{G'_1}(A)| = 1$. Then $d_{G'_1}(A) \geq 4 + |\tau_{G'_1}(A)|$. If $d_{G'_1}(A) \geq 6$, by Proposition 2.2 (2), we have that $d_{G'_1}(A) \geq 4 + |\tau_{G'_1}(A)|$.

Now G'_1 satisfies all the conditions of Lemma 2.4. By Lemma 2.4, G'_1 has a $\beta_{G'_1}$ -orientation extended from the pre-orientation of $E'_1(z_0)$ of all edges incident with z_0 , which implies that G has a β_G -orientation with $\beta_G = 0$. By Proposition 2.3, G has a nowhere-zero 3-flow, which is a contradiction. \square

4 Proof of Theorem 1.17

Assume not. Suppose that G is a counterexample, such that $|V(G)| + |E(G)|$ is as small as possible. Let $S' = \{x \in V(G) : d_G(x) = 5\}$ and $S = \{C = \partial_G(X) : |C| = 5, X \subset V(G)\}$. Let β_G be a Z_3 -boundary, such that G has no β_G -orientation.

Claim 1. $|V(G)| \geq 3$ and $|S'| \leq |S| \leq 5$.

Proof. Since G is 5-edge-connected, $|V(G)| \geq 2$. If $|V(G)| = 2$, let $V(G) = \{x, y\}$. Then all the edges of G are between x and y , and $|E(G)| \geq 5$. Let D_x be an orientation of x , such that $d_{D_x}^+(x) - d_{D_x}^-(x) \equiv \beta_G(x) \pmod{3}$. Since β_G is a Z_3 -boundary, $d_{D_x}^+(y) - d_{D_x}^-(y) \equiv \beta_G(y) \pmod{3}$. Therefore G has a β_G -orientation, a contradiction. Hence $|V(G)| \geq 3$ and $|S'| \leq |S| \leq 5$. \blacksquare

Claim 2. Let $U \subset V(G)$ with $|U| \geq 2$. If $d_G(U) = 5$, then $U \cap S' \neq \emptyset$.

Proof. If not, choose U to be a minimal one such that: for any $A \subset U$ with $2 \leq |A| < |U|$, we have $d_G(A) \neq 5$.

By the minimality of G , G/U has a β_G -orientation D' which is a partial β_G -orientation of G , such that $d_{D'}^+(x) - d_{D'}^-(x) \equiv \beta_G(x) \pmod{3}$ for each $x \in V(G) \setminus U$.

Let G' be a graph obtained from G by contracting $V(G) \setminus U$ as z_0 , and let $\beta_{G'} = \beta_G$.

(i) Since $V(G') = U + z_0$, $|V(G')| = |U| + 1 \geq 3$.

(ii) Since $d_{G'}(z_0) = d_G(U) = 5$, by Proposition 2.2 (1), we have that $d_{G'}(z_0) \leq 4 + |\tau_{G'}(z_0)|$.

(iii) By the assumption and minimality of U , we have that $\forall A \subset U$, $d_G(A) \neq 5$.

Therefore $d_G(A) \geq 6$. By Proposition 2.2 (2), $d_{G'}(A) = d_G(A) \geq 4 + |\tau_{G'}(A)|$.

By Lemma 2.4, the pre-orientation of $E'(z_0)$ of all edges incident with z_0 can be extended to a $\beta_{G'}$ -orientation of G' . Then G has a β_G -orientation, which is a contradiction. \blacksquare

Let G'_1 be a graph obtained from G by adding a new vertex z_0 and $|S'|$ arcs from z_0 to S' , such that each vertex in S' has degree 6 in G'_1 .

Define $\beta_{G'_1}$ as follows:

- (1) $\beta_{G'_1}(x) = \beta_G(x)$ if $x \notin S' + z_0$;
- (2) $\beta_{G'_1}(x) \equiv \beta_G(x) - 1 \pmod{3}$ if $x \in S'$;
- (3) $\beta_{G'_1}(z_0) \equiv |S'| \pmod{3}$ and $\beta_{G'_1}(z_0) \in \{0, 1, 2\}$.

Now $d_{G'_1}(z_0) \leq 4 + |\tau_{G'_1}(z_0)|$ and $|V(G'_1)| = |V(G)| + 1 \geq 4$. We claim that $d_{G'_1}(A) \geq 4 + |\tau_{G'_1}(A)|$, for each nonempty vertex subset A not containing z_0 with $|V(G'_1) \setminus A| > 1$.

If $A \cap S' = \emptyset$, then by Claim 2, $d_{G'_1}(A) = d_G(A) \neq 5$. Thus $d_{G'_1}(A) \geq 6$. By Proposition 2.2 (2), $d_{G'_1}(A) \geq 4 + |\tau_{G'_1}(A)|$.

If $A \cap S' \neq \emptyset$, then $d_{G'_1}(A) \geq d_G(A) + 1 \geq 6$. By Proposition 2.2 (2), we have that $d_{G'_1}(A) \geq 4 + |\tau_{G'_1}(A)|$.

Now G'_1 satisfies all the conditions of Lemma 2.4. By Lemma 2.4, G'_1 has a $\beta_{G'_1}$ -orientation extended from the pre-orientation of $E'_1(z_0)$ of all edges incident with z_0 , which implies that G has a β_G -orientation, a contradiction.

The proof is complete. □

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